

EXISTENCE OF GLOBAL INVARIANT JET DIFFERENTIALS ON PROJECTIVE HYPERSURFACES OF HIGH DEGREE

SIMONE DIVERIO

ABSTRACT. Let $X \subset \mathbb{P}^{n+1}$ be a smooth complex projective hypersurface. In this paper we show that, if the degree of X is large enough, then there exist global sections of the bundle of invariant jet differentials of order n on X , vanishing on an ample divisor. We also prove a logarithmic version, effective in low dimension, for the log-pair (\mathbb{P}^n, D) , where D is a smooth irreducible divisor of high degree. Moreover, these result are sharp, *i.e.* one cannot have such jet differentials of order less than n . Kobayashi hyperbolicity, invariant jet differentials, algebraic holomorphic Morse inequalities, complex projective hypersurfaces, logarithmic variety, logarithmic jet bundle, Schur power.

1. INTRODUCTION

Let X be a compact complex manifold. According to a well-know criterion of Brody, X is Kobayashi hyperbolic if and only if there are no non-constant entire holomorphic curves in X . In 1970, S. Kobayashi [Kob70] conjectured that if $X \subset \mathbb{P}^{n+1}$ is a generic hypersurface of degree $d = \deg X$ at least equal to $2n + 1$, then X is Kobayashi hyperbolic (analogously, he proposed also the following logarithmic version of his conjecture: if $D \subset \mathbb{P}^n$ is a generic irreducible divisor of degree $\deg D \geq 2n + 1$, then $\mathbb{P}^n \setminus D$ is Kobayashi hyperbolic). Thus proving the “compact” Kobayashi conjecture is equivalent to proving the non existence of entire holomorphic curves on a generic projective hypersurface of degree large enough.

Several decades after the pioneering work of Bloch in 1926, it has been realized that an essential tool for controlling the geometry of entire curves on a manifold X is to produce differential equations on X that every entire curve must satisfy. For instance, in 1979, Green and Griffiths [G-G79] constructed the sheaf $\mathcal{J}_{k,m}$ of jet differentials of order k and weighted degree m and were able to prove the Bloch conjecture (*i.e.*, that every entire holomorphic curve in a projective variety is algebraically degenerate as soon as the irregularity is greater than the dimension).

Several years later, Siu outlined new ideas for proving Kobayashi’s conjecture, by making use of jet differentials and by generalizing some techniques due to Clemens, Ein and Voisin (see [Siu04]). However many details are missing, and it also seems to be hard to derive effective results from Siu’s approach.

We would like here to concentrate on a refined and more geometrical version of the bundle of Green and Griffiths, namely the bundle of *invariant jet differentials*, which was first introduced in this context by Demailly in

[Dem95]. This bundle reflects better the geometry of entire curves since it just takes care of the image of such curves and not of the way they are parametrized: following [Dem95], we will denote it $E_{k,m}T_X^*$. The general philosophy is that global holomorphic sections of $E_{k,m}T_X^*$ vanishing on a fixed ample divisor give rise to global algebraic differential equations that every entire holomorphic curve must satisfy.

It is known by [D-EG00] that every smooth surface in \mathbb{P}^3 of degree greater or equal to 15 has such differential equations of order two. For the dimension three case, Rousseau [Rou06] observed that one needs to look for order three equations since one has in general the vanishing of symmetric differentials and invariant 2-jet differentials for smooth hypersurfaces in projective 4-space. On the other hand [Rou06] shows the existence of global invariant 3-jet differentials vanishing on an ample divisor on every smooth hypersurface X in \mathbb{P}^4 , provided that $\deg X \geq 97$.

Recently, in [Div08], we improved the bound for the degree obtained in [Rou06] and found the existence of invariant jet differentials for smooth projective hypersurfaces of dimension at most 8 (with an explicit effective lower bound for the degree of the hypersurface up to dimension 5).

Until our paper [Div08], the existence was obtained by showing first that the Euler characteristic $\chi(E_{k,m}T_X^*)$ of the bundle of invariant jet differentials is positive for m large enough. Then, with a delicate study of the even cohomology groups of such bundles – which usually involves the rather difficult investigation of the composition series of $E_{k,m}T_X^*$ – one could obtain in principle a positive lower bound for $h^0(X, E_{k,m}T_X^*)$ in terms of the Euler characteristic.

Here we generalize the result of [Div08] to arbitrary dimension, thus solving the problem of finding *invariant* jet differentials on complex projective hypersurfaces of high degree. Namely we get the following.

Theorem 1.1. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth complex projective hypersurface and let $A \rightarrow X$ be an ample line bundle. Then there exists a positive integer δ_n such that*

$$H^0(X, E_{k,m}T_X^* \otimes A^{-1}) \neq 0, \quad k \geq n,$$

provided that $\deg(X) \geq \delta_n$ and m is large enough.

In other words, on every smooth n -dimensional complex projective hypersurface of sufficiently high degree, there exist global invariant jet differentials of order n vanishing on an ample divisor, and every entire curve must satisfy the corresponding differential equation.

Unfortunately, the lower bound for the degree of X is effective just theoretically, and one can compute explicit values just for low dimensions (see [Div08]). Nevertheless, the result is sharp as far as the order k of jets is concerned since, by a theorem of [Div08], there are no jet differentials of order $k < n$ on a smooth projective hypersurface of dimension n .

We also prove a logarithmic version of the above theorem.

Theorem 1.2. *Let $D \subset \mathbb{P}^n$ be a smooth irreducible divisor and let $A \rightarrow \mathbb{P}^n$ be an ample line bundle. Then there exists a positive integer δ_n such that*

$$H^0(\mathbb{P}^n, E_{k,m}T_{\mathbb{P}^n}^* \langle D \rangle \otimes A^{-1}) \neq 0, \quad k \geq n,$$

TABLE 1. Effective lower bound for degree δ .

n	k				
	1	2	3	4	5
2		15	14	14	14
3			75	67	67
4				306	280
5					1154

provided that $\deg(D) \geq \delta_n$ and m is large enough.

Moreover, we have the effective lower bounds for the degree δ of D as shown in Table 1 (depending on the values of n and k).

In the statement above, the bundle $E_{k,m}T_{\mathbb{P}^n}^*\langle D \rangle$, is the vector bundle of logarithmic invariant jet differentials, introduced in the general setting of logarithmic varieties by Dethloff and Lu in [De-L01].

Finally, we would like to stress that our proof is based on the algebraic version of holomorphic Morse inequalities of [Tra95], and so we deal directly with the dimension of the space of global sections: we are able in this way to skip entirely the arduous study of the higher cohomology and of the graded bundle associated to $E_{k,m}T_X^*$.

Acknowledgements. We are indebted to Prof. Stefano Trapani, who suggested to us the main ideas in Lemma 3.2 and Lemma 3.3.

We would like to thank also Prof. Jean-Pierre Demailly for a careful reading of the present manuscript, and for many useful suggestions and comments.

Finally, thanks to Gianluca Pacienza who suggested us to try to extend our results to the logarithmic case.

2. NOTATIONS AND PRELIMINARY MATERIAL

Let X be a compact complex manifold and $V \subset T_X$ a holomorphic (non necessarily integrable) subbundle of the tangent bundle of X .

2.1. Invariant jet differentials. The bundle $J_k V \rightarrow X$ is the bundle of k -jets of holomorphic curves $f: (\mathbb{C}, 0) \rightarrow X$ which are tangent to V , i.e., such that $f'(t) \in V_{f(t)}$ for all t in a neighbourhood of 0, together with the projection map $f \mapsto f(0)$ onto X .

Let \mathbb{G}_k be the group of germs of k -jets of biholomorphisms of $(\mathbb{C}, 0)$, that is, the group of germs of biholomorphic maps

$$t \mapsto \varphi(t) = a_1 t + a_2 t^2 + \cdots + a_k t^k, \quad a_1 \in \mathbb{C}^*, a_j \in \mathbb{C}, j \geq 2,$$

in which the composition law is taken modulo terms t^j of degree $j > k$. Then \mathbb{G}_k admits a natural fiberwise right action on $J_k V$ consisting of reparametrizing k -jets of curves by a biholomorphic change of parameter.

Next, we define the bundle of Demailly-Semple jet differentials (or invariant jet differentials).

Definition 2.1 ([Dem95]). The vector bundle of *invariant jet differentials of order k and degree m* is the bundle $E_{k,m}V^* \rightarrow X$ of polynomial differential operators $Q(f', f'', \dots, f^{(k)})$ over the fibers of $J_k V$, which are invariant under arbitrary changes of parametrization, *i.e.* for every $\varphi \in \mathbb{G}_k$

$$Q((f \circ \varphi)', (f \circ \varphi)'', \dots, (f \circ \varphi)^{(k)}) = \varphi'(0)^m Q(f', f'', \dots, f^{(k)}).$$

2.2. Projectivized jet bundles. Here is the construction of the tower of projectivized bundles which provides a (relative) smooth compactification of $J_k^{\text{reg}}V/\mathbb{G}_k$, where $J_k^{\text{reg}}V$ is the bundle of regular k -jets tangent to V , that is k -jets such that $f'(0) \neq 0$.

Let $\dim X = n$ and $\text{rank } V = r$. With (X, V) we associate another “directed” manifold (\tilde{X}, \tilde{V}) where $\tilde{X} = P(V)$ is the projectivized bundle of lines of V , $\pi: \tilde{X} \rightarrow X$ is the natural projection and \tilde{V} is the subbundle of $T_{\tilde{X}}$ defined fiberwise as

$$\tilde{V}_{(x_0, [v_0])} \stackrel{\text{def}}{=} \{\xi \in T_{\tilde{X}, (x_0, [v_0])} \mid \pi_* \xi \in \mathbb{C} \cdot v_0\},$$

$x_0 \in X$ and $v_0 \in T_{X, x_0} \setminus \{0\}$. We also have a “lifting” operator which assigns to a germ of holomorphic curve $f: (\mathbb{C}, 0) \rightarrow X$ tangent to V a germ of holomorphic curve $\tilde{f}: (\mathbb{C}, 0) \rightarrow \tilde{X}$ tangent to \tilde{V} in such a way that $\tilde{f}(t) = (f(t), [f'(t)])$.

To construct the projectivized k -jet bundle we simply set inductively $(X_0, V_0) = (X, V)$ and $(X_k, V_k) = (\tilde{X}_{k-1}, \tilde{V}_{k-1})$. Of course, we have for each $k > 0$ a tautological line bundle $\mathcal{O}_{X_k}(-1) \rightarrow X_k$ and a natural projection $\pi_k: X_k \rightarrow X_{k-1}$. We shall call $\pi_{j,k}$ the composition of the projections $\pi_{j+1} \circ \dots \circ \pi_k$, so that the total projection is given by $\pi_{0,k}: X_k \rightarrow X$. For each $k > 0$, we have short exact sequences

$$(1) \quad 0 \rightarrow T_{X_k/X_{k-1}} \rightarrow V_k \rightarrow \mathcal{O}_{X_k}(-1) \rightarrow 0,$$

$$(2) \quad 0 \rightarrow \mathcal{O}_{X_k} \rightarrow \pi_k^* V_{k-1} \otimes \mathcal{O}_{X_k}(1) \rightarrow T_{X_k/X_{k-1}} \rightarrow 0,$$

where $T_{X_k/X_{k-1}} = \ker(\pi_k)_*$ is the vertical tangent bundle relative to π_k and $\text{rank } V_k = r$, $\dim X_k = n + k(r-1)$. Here, we also have an inductively defined k -lifting for germs of holomorphic curves such that $f_{[k]}: (\mathbb{C}, 0) \rightarrow X_k$ is obtained as $f_{[k]} = \tilde{f}_{[k-1]}$.

The following theorem is the link between these projectivized bundles and jet differentials.

Theorem 2.1 ([Dem95]). *Suppose that $\text{rank } V \geq 2$. The quotient space $J_k^{\text{reg}}V/\mathbb{G}_k$ has the structure of a locally trivial bundle over X , and there is a holomorphic embedding $J_k^{\text{reg}}V/\mathbb{G}_k \hookrightarrow X_k$ over X , which identifies $J_k^{\text{reg}}V/\mathbb{G}_k$ with X_k^{reg} , that is the set of point in X_k on the form $f_{[k]}(0)$ for some non singular k -jet f . In other words X_k is a relative compactification of $J_k^{\text{reg}}V/\mathbb{G}_k$ over X .*

Moreover, we have the direct image formula

$$(\pi_{0,k})_* \mathcal{O}_{X_k}(m) = \mathcal{O}(E_{k,m}V^*).$$

Next, here is the link between the theory of hyperbolicity and invariant jet differentials.

Theorem 2.2 ([G-G79],[Dem95]). *Assume that there exist integers $k, m > 0$ and an ample line bundle $A \rightarrow X$ such that*

$$H^0(X_k, \mathcal{O}_{X_k}(m) \otimes \pi_{0,k}^* A^{-1}) \simeq H^0(X, E_{k,m} V^* \otimes A^{-1})$$

has non zero sections $\sigma_1, \dots, \sigma_N$ and let $Z \subset X_k$ be the base locus of these sections. Then every entire holomorphic curve $f: \mathbb{C} \rightarrow X$ tangent to V is such that $f_{[k]}(\mathbb{C}) \subset Z$. In other words, for every global \mathbb{G}_k -invariant differential equation P vanishing on an ample divisor, every entire holomorphic curve f must satisfy the algebraic differential equation $P(f) = 0$.

2.3. Cohomology ring of X_k . Denote by $c_\bullet(E)$ the total Chern class of a vector bundle E . The short exact sequences (1) and (2) give, for each $k > 0$, the following formulae:

$$c_\bullet(V_k) = c_\bullet(T_{X_k/X_{k-1}})c_\bullet(\mathcal{O}_{X_k}(-1))$$

and

$$c_\bullet(\pi_k^* V_{k-1} \otimes \mathcal{O}_{X_k}(1)) = c_\bullet(T_{X_k/X_{k-1}}),$$

so that

$$(3) \quad c_\bullet(V_k) = c_\bullet(\mathcal{O}_{X_k}(-1))c_\bullet(\pi_k^* V_{k-1} \otimes \mathcal{O}_{X_k}(1)).$$

Let us call $u_j = c_1(\mathcal{O}_{X_j}(1))$ and $c_l^{[j]} = c_l(V_j)$. With these notations, (3) becomes

$$(4) \quad c_l^{[k]} = \sum_{s=0}^l \left[\binom{n-s}{l-s} - \binom{n-s}{l-s-1} \right] u_k^{l-s} \cdot \pi_k^* c_s^{[k-1]}, \quad 1 \leq l \leq r.$$

Since X_j is the projectivized bundle of line of V_{j-1} , we also have the polynomial relations

$$(5) \quad u_j^r + \pi_j^* c_1^{[j-1]} \cdot u_j^{r-1} + \dots + \pi_j^* c_{r-1}^{[j-1]} \cdot u_j + \pi_j^* c_r^{[j-1]} = 0, \quad 1 \leq j \leq k.$$

After all, the cohomology ring of X_k is defined in terms of generators and relations as the polynomial algebra $H^\bullet(X)[u_1, \dots, u_k]$ with the relations (5) in which, of course, utilizing inductively (4), we have that $c_l^{[j]}$ is a polynomial with integral coefficients in the variables $u_1, \dots, u_j, c_1(V), \dots, c_l(V)$.

In particular, for the first Chern class of V_k , we obtain the very simple expression

$$(6) \quad c_1^{[k]} = \pi_{0,k}^* c_1(V) + (r-1) \sum_{s=1}^k \pi_{s,k}^* u_s.$$

2.4. Algebraic holomorphic Morse inequalities. Let $L \rightarrow X$ be a holomorphic line bundle over a compact Kähler manifold of dimension n and $E \rightarrow X$ a holomorphic vector bundle of rank r . Suppose that L can be written as the difference of two nef line bundles, say $L = F \otimes G^{-1}$, with $F, G \rightarrow X$ numerically effective. Then we have the following asymptotic estimate for the partial alternating sum of the dimension of cohomology groups of powers of L with values in E .

Theorem 2.3 ([Dem00]). *With the previous notation, we have (strong algebraic holomorphic Morse inequalities) :*

$$\sum_{j=0}^q (-1)^{q-j} h^j(X, L^{\otimes m} \otimes E) \leq r \frac{m^n}{n!} \sum_{j=0}^q (-1)^{q-j} \binom{n}{j} F^{n-j} \cdot G^j + o(m^n).$$

In particular [Tra95], $L^{\otimes m} \otimes E$ has a global section for m large as soon as $F^n - n F^{n-1} \cdot G > 0$.

2.5. A vanishing theorem. Let $X \subset \mathbb{P}^N$ be a smooth complete intersection of dimension $\dim X = n$. In [Div08], we proved the following vanishing theorem for the space of global sections of invariant jet differentials:

Theorem 2.4 ([Div08]). *Let $X \subset \mathbb{P}^N$ be a smooth complete intersection of dimension $\dim X = n$. Then*

$$H^0(X, E_{k,m} T_X^*) = 0$$

for all $m \geq 1$ and $1 \leq k < n/(N-n)$. In particular, if X is a smooth projective hypersurface, then

$$H^0(X, E_{k,m} T_X^*) = 0$$

for all $m \geq 1$ and $1 \leq k \leq n-1$.

3. PROOF OF THEOREM 1.1

The idea of the proof is to apply the algebraic holomorphic Morse inequalities to a particular relatively nef line bundle over X_n which admits a nontrivial morphism to (a power of) $\mathcal{O}_{X_n}(1)$ and then to conclude by the direct image argument of Theorem 2.1.

3.1. Sufficient conditions for relative nefness. By definition, there is a canonical injection $\mathcal{O}_{X_k}(-1) \hookrightarrow \pi_k^* V_{k-1}$ and a composition with the differential of the projection $(\pi_k)_*$ yields, for all $k \geq 2$, a canonical line bundle morphism

$$\mathcal{O}_{X_k}(-1) \hookrightarrow \pi_k^* V_{k-1} \rightarrow \pi_k^* \mathcal{O}_{X_{k-1}}(-1),$$

which admits precisely $D_k \stackrel{\text{def}}{=} P(T_{X_{k-1}/X_{k-2}}) \subset P(V_{k-1}) = X_k$ as its zero divisor. Hence, we find

$$(7) \quad \mathcal{O}_{X_k}(1) = \pi_k^* \mathcal{O}_{X_{k-1}}(1) \otimes \mathcal{O}(D_k).$$

Now, for $\mathbf{a} = (a_1, \dots, a_k) \in \mathbb{Z}^k$, define a line bundle $\mathcal{O}_{X_k}(\mathbf{a})$ on X_k as

$$\mathcal{O}_{X_k}(\mathbf{a}) = \pi_{1,k}^* \mathcal{O}_{X_1}(a_1) \otimes \pi_{2,k}^* \mathcal{O}_{X_2}(a_2) \otimes \dots \otimes \mathcal{O}_{X_k}(a_k).$$

By (7), we have

$$\pi_{j,k}^* \mathcal{O}_{X_j}(1) = \mathcal{O}_{X_k}(1) \otimes \mathcal{O}_{X_k}(-\pi_{j+1,k}^* D_{j+1} - \dots - D_k),$$

thus by putting $D_j^* = \pi_{j+1,k}^* D_{j+1}$ for $j = 1, \dots, k-1$ and $D_k^* = 0$, we have an identity

$$\mathcal{O}_{X_k}(\mathbf{a}) = \mathcal{O}_{X_k}(b_k) \otimes \mathcal{O}_{X_k}(-\mathbf{b} \cdot D^*), \quad \text{where}$$

$$\mathbf{b} = (b_1, \dots, b_k) \in \mathbb{Z}^k, \quad b_j = a_1 + \dots + a_j,$$

$$\mathbf{b} \cdot D^* = \sum_{j=1}^{k-1} b_j \pi_{j+1,k}^* D_{j+1}.$$

In particular, if $\mathbf{b} \in \mathbb{N}^k$, that is if $a_1 + \dots + a_j \geq 0$, we get a nontrivial morphism

$$\mathcal{O}_{X_k}(\mathbf{a}) = \mathcal{O}_{X_k}(b_k) \otimes \mathcal{O}_{X_k}(-\mathbf{b} \cdot D^*) \rightarrow \mathcal{O}_{X_k}(b_k).$$

We then have the following:

Proposition 3.1 ([Dem95]). *Let $\mathbf{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$ and $m = a_1 + \dots + a_k$.*

- *We have the direct image formula*

$$(\pi_{0,k})_* \mathcal{O}_{X_k}(\mathbf{a}) \simeq \mathcal{O}(\overline{F}^{\mathbf{a}} E_{k,m} V^*) \subset \mathcal{O}(E_{k,m} V^*)$$

where $\overline{F}^{\mathbf{a}} E_{k,m} V^*$ is the subbundle of polynomials $Q(f', \dots, f^{(k)})$ of $E_{k,m} V^*$ involving only monomials $(f^{(\bullet)})^\ell$ such that

$$\ell_{s+1} + 2\ell_{s+2} + \dots + (k-s)\ell_k \leq a_{s+1} + \dots + a_k$$

for all $s = 0, \dots, k-1$.

- *If*

$$(8) \quad a_1 \geq 3a_2, \dots, a_{k-2} \geq 3a_{k-1} \quad \text{and} \quad a_{k-1} \geq 2a_k > 0,$$

the line bundle $\mathcal{O}_{X_k}(\mathbf{a})$ is relatively nef over X .

From now on, we will set in the absolute case, that is $V = T_X$. So, let $X \subset \mathbb{P}^{n+1}$ be a smooth complex hypersurface of degree $\deg X = d$.

Now, for the projective hypersurfaces case, it is always possible to express $\mathcal{O}_{X_k}(\mathbf{a})$ as the difference of two globally nef line bundles, provided condition (8) is satisfied. We prove this fact in the next:

Proposition 3.2. *Let $X \subset \mathbb{P}^{n+1}$ be a smooth projective hypersurface and $\mathcal{O}_X(1)$ be the hyperplane divisor on X . If condition (8) holds, then $\mathcal{O}_{X_k}(\mathbf{a}) \otimes \pi_{0,k}^* \mathcal{O}_X(\ell)$ is nef provided that $\ell \geq 2|\mathbf{a}|$, where $|\mathbf{a}| = a_1 + \dots + a_k$.*

In particular $\mathcal{O}_{X_k}(\mathbf{a}) = (\mathcal{O}_{X_k}(\mathbf{a}) \otimes \pi_{0,k}^ \mathcal{O}_X(2|\mathbf{a}|)) \otimes \pi_{0,k}^* \mathcal{O}_X(-2|\mathbf{a}|)$ and both $\mathcal{O}_{X_k}(\mathbf{a}) \otimes \pi_{0,k}^* \mathcal{O}_X(2|\mathbf{a}|)$ and $\pi_{0,k}^* \mathcal{O}_X(2|\mathbf{a}|)$ are nef.*

Proof. In [Div08] we proved that the line bundle

$$\mathcal{O}_{X_k}(2 \cdot 3^{k-2}, 2 \cdot 3^{k-3}, \dots, 6, 2, 1) \otimes \pi_{0,k}^* \mathcal{O}_X(\ell)$$

is nef as soon as $\ell \geq 2 \cdot (1 + 2 + 6 + \dots + 2 \cdot 3^{k-2}) = 2 \cdot 3^{k-1}$. Now we take $\mathbf{a} = (a_1, \dots, a_k) \in \mathbb{N}^k$ such that $a_1 \geq 3a_2, \dots, a_{k-2} \geq 3a_{k-1}, a_{k-1} \geq 2a_k > 0$ and we proceed by induction, the case $k = 1$ being obvious. Write

$$\begin{aligned} & \mathcal{O}_{X_k}(a_1, a_2, \dots, a_k) \otimes \pi_{0,k}^* \mathcal{O}_X(2 \cdot (a_1 + \dots + a_k)) \\ &= (\mathcal{O}_{X_k}(2 \cdot 3^{k-2}, \dots, 6, 2, 1) \otimes \pi_{0,k}^* \mathcal{O}_X(2 \cdot 3^{k-1}))^{\otimes a_k} \\ & \quad \otimes \pi_k^* \left(\mathcal{O}_{X_{k-1}}(a_1 - 2 \cdot 3^{k-2} a_k, \dots, a_{k-2} - 6a_k, a_{k-1} - 2a_k) \right. \\ & \quad \left. \otimes \pi_{0,k-1}^* \mathcal{O}_X(2 \cdot (a_1 + \dots + a_k - 3^{k-1} a_k)) \right). \end{aligned}$$

Therefore, we have to prove that

$$\begin{aligned} & \mathcal{O}_{X_{k-1}}(a_1 - 2 \cdot 3^{k-2} a_k, \dots, a_{k-2} - 6a_k, a_{k-1} - 2a_k) \\ & \quad \otimes \pi_{0,k-1}^* \mathcal{O}_X(2 \cdot (a_1 + \dots + a_k - 3^{k-1} a_k)) \end{aligned}$$

is nef. Our chain of inequalities gives, for $1 \leq j \leq k-2$, $a_j \geq 3^{k-j-1}a_k$ and $a_{k-1} \geq 2a_k$. Thus, condition (8) is satisfied by the weights of

$$\mathcal{O}_{X_{k-1}}(a_1 - 2 \cdot 3^{k-2}a_k, \dots, a_{k-2} - 6a_k, a_{k-1} - 2a_k)$$

and $2 \cdot (a_1 + \dots + a_k - 3^{k-1}a_k)$ is exactly twice the sum of these weights. \square

Remark 1. At this point it should be clear that to prove our Theorem, it is sufficient to show the existence of an n -tuple (a_1, \dots, a_n) satisfying condition (8) and such that

$$(9) \quad (\mathcal{O}_{X_n}(\mathbf{a}) \otimes \pi_{0,n}^* \mathcal{O}_X(2|\mathbf{a}|))^{n^2} - n^2 (\mathcal{O}_{X_n}(\mathbf{a}) \otimes \pi_{0,n}^* \mathcal{O}_X(2|\mathbf{a}|))^{n^2-1} \cdot \pi_{0,n}^* \mathcal{O}_X(2|\mathbf{a}|) > 0$$

for $d = \deg X$ large enough, where $n^2 = n + n(n-1) = \dim X_n$.

In fact, this would show the bigness of $\mathcal{O}_{X_n}(\mathbf{a}) \hookrightarrow \mathcal{O}_{X_n}(|\mathbf{a}|)$ and so the bigness of $\mathcal{O}_{X_n}(1)$.

3.2. Evaluation in terms of the degree. For $X \subset \mathbb{P}^{n+1}$ a smooth projective hypersurface of degree $\deg X = d$, standard arguments involving the Euler exact sequence, show that

$$c_j(X) = c_j(T_X) = h^j((-1)^j d^j + o(d^j)), \quad j = 1, \dots, n,$$

where $h \in H^2(X, \mathbb{Z})$ is the hyperplane class and $o(d^j)$ is a polynomial in d of degree at most $j-1$.

Proposition 3.3. *The quantities*

$$(\mathcal{O}_{X_k}(\mathbf{a}) \otimes \pi_{0,k}^* \mathcal{O}_X(2|\mathbf{a}|))^{n+k(n-1)} - [n+k(n-1)](\mathcal{O}_{X_k}(\mathbf{a}) \otimes \pi_{0,k}^* \mathcal{O}_X(2|\mathbf{a}|))^{n+k(n-1)-1} \cdot \pi_{0,k}^* \mathcal{O}_X(2|\mathbf{a}|)$$

and

$$\mathcal{O}_{X_k}(\mathbf{a})^{n+k(n-1)}$$

are both polynomials in the variable d with coefficients in $\mathbb{Z}[a_1, \dots, a_k]$ of degree at most $n+1$ and the coefficients of d^{n+1} of the two expressions are equal.

Moreover this coefficient is a homogeneous polynomial in a_1, \dots, a_k of degree $n+k(n-1)$ or identically zero.

Proof. Set $\mathcal{F}_k(\mathbf{a}) = \mathcal{O}_{X_k}(\mathbf{a}) \otimes \pi_{0,k}^* \mathcal{O}_X(2|\mathbf{a}|)$ and $\mathcal{G}_k(\mathbf{a}) = \pi_{0,k}^* \mathcal{O}_X(2|\mathbf{a}|)$. Then we have

$$\begin{aligned} & \mathcal{F}_k(\mathbf{a})^{n+k(n-1)} + [n+k(n-1)]\mathcal{F}_k(\mathbf{a})^{n+k(n-1)-1} \cdot \mathcal{G}_k(\mathbf{a}) \\ &= \mathcal{O}_{X_k}(\mathbf{a})^{n+k(n-1)} + \text{terms which have } \mathcal{G}_k(\mathbf{a}) \text{ as a factor.} \end{aligned}$$

Now we use relations (4) and (5) to observe that

$$\mathcal{O}_{X_k}(\mathbf{a})^{n+k(n-1)} = \sum_{j_1+2j_2+\dots+nj_n=n} P_{j_1 \dots j_n}^{[k]}(\mathbf{a}) c_1(X)^{j_1} \dots c_n(X)^{j_n},$$

where the $P_{j_1 \dots j_n}^{[k]}(\mathbf{a})$'s are homogeneous polynomial of degree $n+k(n-1)$ in the variables a_1, \dots, a_k (or possibly identically zero). Thus, substituting

the $c_j(X)$'s with their expression in terms of the degree, we get

$$\mathcal{O}_{X_k}(\mathbf{a})^{n+k(n-1)} = (-1)^n \left(\sum_{j_1+2j_2+\dots+nj_n=n} P_{j_1\dots j_n}^{[k]}(\mathbf{a}) \right) d^{n+1} + o(d^{n+1}),$$

since $h^n = d$. On the other hand, using relations (4) and (5) on terms which have $\mathcal{G}_k(\mathbf{a})$ as a factor, gives something of the form

$$\sum_{\substack{j_1+2j_2+\dots+nj_n+i=n \\ i>0}} Q_{j_1\dots j_n i}^{[k]}(\mathbf{a}) h^i \cdot c_1(X)^{j_1} \dots c_n(X)^{j_n},$$

since $c_1(\mathcal{G}_k(\mathbf{a})) = 2|\mathbf{a}|h$ and $\mathcal{G}_k(\mathbf{a})$ is always a factor. Substituting the $c_j(X)$'s with their expression in terms of the degree, we get here

$$h^i \cdot c_1(X)^{j_1} \dots c_n(X)^{j_n} = (-1)^{j_1+\dots+j_n} \underbrace{h^n}_{=d} \cdot d^{j_1+\dots+j_n} = o(d^{n+1}).$$

□

We need a lemma.

Lemma 3.1. *Let $\mathfrak{C} \subset \mathbb{R}^k$ be a cone with nonempty interior. Let $\mathbb{Z}^k \subset \mathbb{R}^k$ be the canonical lattice in \mathbb{R}^k . Then $\mathbb{Z}^k \cap \mathfrak{C}$ is Zariski dense in \mathbb{R}^k .*

Proof. Since \mathfrak{C} is a cone with nonempty interior, it contains cubes of arbitrary large edges, so $\mathbb{Z}^k \cap \mathfrak{C}$ contains a product of integer intervals $\prod [\alpha_i, \beta_i]$ with $\beta_i - \alpha_i > N$. By using induction on dimension, this implies that a polynomial P of degree at most N vanishing on $\mathbb{Z}^k \cap \mathfrak{C}$ must be identically zero. As N can be taken arbitrary large, we conclude that $\mathbb{Z}^k \cap \mathfrak{C}$ is Zariski dense. □

Corollary 3.1. *If the top self-intersection $\mathcal{O}_{X_k}(\mathbf{a})^{n+k(n-1)}$ has degree exactly equal to $n+1$ in d for some choice of \mathbf{a} , then $\mathcal{O}_{X_k}(m) \otimes \pi_{0,k}^* A^{-1}$ has a global section for all line bundle $A \rightarrow X$ and for all d, m sufficiently large.*

Proof. The real k -tuples which satisfy condition (8), form a cone with nonempty interior in \mathbb{R}^k . Thus, by Lemma 3.1, there exists an integral \mathbf{a}' satisfying condition (8) and such that $\mathcal{O}_{X_k}(\mathbf{a}')^{n+k(n-1)}$ has degree exactly $n+1$ in d . For reasons similar to those in the proof of Proposition 3.3, the coefficient of degree $n+1$ in d of $\mathcal{O}_{X_k}(\mathbf{a}')^{n+k(n-1)}$ and $(\mathcal{O}_{X_k}(\mathbf{a}') \otimes \pi_{0,k}^* \mathcal{O}_X(2|\mathbf{a}'|))^{n+k(n-1)}$ are the same; the second one being nef, this coefficient must be positive.

Now, by Proposition 3.3, this coefficient is the same as the coefficient of degree $n+1$ in d of

$$\begin{aligned} & (\mathcal{O}_{X_k}(\mathbf{a}') \otimes \pi_{0,k}^* \mathcal{O}_X(2|\mathbf{a}'|))^{n+k(n-1)} \\ & - [n+k(n-1)] (\mathcal{O}_{X_k}(\mathbf{a}') \otimes \pi_{0,k}^* \mathcal{O}_X(2|\mathbf{a}'|))^{n+k(n-1)-1} \cdot \pi_{0,k}^* \mathcal{O}_X(2|\mathbf{a}'|). \end{aligned}$$

But then this last quantity is positive for d large enough, and the Corollary follows by an application of algebraic holomorphic Morse inequalities. □

Corollary 3.2. *For $k < n$, the coefficient of d^{n+1} in the expression of*

$$\mathcal{O}_{X_k}(\mathbf{a})^{n+k(n-1)}$$

is identically zero.

Proof. Otherwise, we would have global sections of $\mathcal{O}_{X_k}(m)$ for m large and $k < n$, which is impossible by Theorem 2.4. \square

3.3. Bigness of $\mathcal{O}_{X_n}(1)$. Thanks to the results of the previous subsection, to show the existence of a global section of $\mathcal{O}_{X_n}(m) \otimes \pi_{0,n}^* A^{-1}$ for m and d large, we just need to show that $\mathcal{O}_{X_n}(\mathbf{a})^{n^2}$ has degree exactly $n+1$ in d for some n -tuple (a_1, \dots, a_n) .

The multinomial theorem gives

$$\begin{aligned} & (a_1 \pi_{1,k}^* u_1 + \dots + a_k u_k)^{n+k(n-1)} \\ &= \sum_{j_1 + \dots + j_k = n+k(n-1)} \frac{(n+k(n-1))!}{j_1! \dots j_k!} a_1^{j_1} \dots a_k^{j_k} \pi_{1,k}^* u_1^{j_1} \dots u_k^{j_k}. \end{aligned}$$

We need two lemmas.

Lemma 3.2. *The coefficient of degree $n+1$ in d of the two following intersections is zero:*

- $\pi_{1,k}^* u_1^{j_1} \cdot \pi_{2,k}^* u_2^{j_2} \dots u_k^{j_k}$ for all $1 \leq k \leq n-1$ and $j_1 + \dots + j_k = n+k(n-1)$
- $\pi_{1,n-i-1}^* u_1^{j_1} \cdot \pi_{2,n-i-1}^* u_2^{j_2} \dots u_{n-i-1}^{j_{n-i-1}} \cdot \pi_{0,n-i-1}^* c_1(X)^i$ for all $1 \leq i \leq n-2$ and $j_1 + \dots + j_{n-i-1} = (n-i-1)n+1$.

Proof. The first statement is straightforward. By Corollary 3.2, we know that the coefficient of degree $n+1$ in d (once the expression is reduced in term of the degree of the hypersurface) of $(a_1 \pi_{1,k}^* u_1 + \dots + a_k u_k)^{n+k(n-1)}$ must be identically zero for $k < n$. If this first part of the lemma fails to be true for some (j_1, \dots, j_k) , then this leading coefficient would contain at least a monomial, namely $a_1^{j_1} \dots a_k^{j_k}$ and thus it would not be identically zero.

For the second statement we proceed by induction on i . Let us start with $i=1$. By the first part of the present lemma, we have that

$$\pi_{1,n-1}^* u_1^{j_1} \cdot \pi_{2,n-1}^* u_2^{j_2} \dots \pi_{n-1}^* u_{n-2}^{j_{n-2}} \cdot u_{n-1}^n = o(d^{n+1}).$$

On the other hand, relation (5) gives

$$\begin{aligned} & \pi_{1,n-1}^* u_1^{j_1} \cdot \pi_{2,n-1}^* u_2^{j_2} \dots \pi_{n-1}^* u_{n-2}^{j_{n-2}} \cdot u_{n-1}^n \\ &= \pi_{1,n-1}^* u_1^{j_1} \cdot \pi_{2,n-1}^* u_2^{j_2} \dots \pi_{n-1}^* u_{n-2}^{j_{n-2}} \\ & \quad \cdot \left(-\pi_{n-1}^* c_1^{[n-2]} \cdot u_{n-1}^{n-1} - \dots - \pi_{n-1}^* c_{n-1}^{[n-2]} \cdot u_{n-1} - \pi_{n-1}^* c_n^{[n-2]} \right) \\ &= -\pi_{1,n-1}^* u_1^{j_1} \cdot \pi_{2,n-1}^* u_2^{j_2} \dots \pi_{n-1}^* u_{n-2}^{j_{n-2}} \cdot \pi_{n-1}^* c_1^{[n-2]} \cdot u_{n-1}^{n-1} \end{aligned}$$

and the second equality is true for degree reasons:

$$u_1^{j_1} \cdot u_2^{j_2} \dots u_{n-2}^{j_{n-2}} \cdot c_l^{[n-2]}, \quad l = 2, \dots, n,$$

“lives” on X_{n-2} and has total degree $n+(n-2)(n-1)-1+l$ which is strictly greater than $n+(n-2)(n-1) = \dim X_{n-2}$, so that $u_1^{j_1} \cdot u_2^{j_2} \dots u_{n-2}^{j_{n-2}} \cdot c_l^{[n-2]} = 0$.

Now, we use relation (6) and obtain in this way

$$\begin{aligned}
& \pi_{1,n-1}^* u_1^{j_1} \cdot \pi_{2,n-1}^* u_2^{j_2} \cdots \pi_{n-1}^* u_{n-2}^{j_{n-2}} \cdot u_{n-1}^n \\
&= -\pi_{1,n-1}^* u_1^{j_1} \cdot \pi_{2,n-1}^* u_2^{j_2} \cdots \pi_{n-1}^* u_{n-2}^{j_{n-2}} \cdot u_{n-1}^{n-1} \\
&\quad \cdot \left(\pi_{0,n-1}^* c_1(X) + (n-1) \sum_{s=1}^{n-2} \pi_{s,n-1}^* u_s \right) \\
&= -\pi_{1,n-1}^* u_1^{j_1} \cdot \pi_{2,n-1}^* u_2^{j_2} \cdots \pi_{n-1}^* u_{n-2}^{j_{n-2}} \cdot u_{n-1}^{n-1} \cdot \pi_{0,n-1}^* c_1(X) \\
&\quad - (n-1) u_{n-1}^{n-1} \cdot \sum_{s=1}^{n-2} \pi_{1,n-1}^* u_1^{j_1} \cdots \pi_{s,n-1}^* u_s^{j_s+1} \cdots \pi_{n-1}^* u_{n-2}^{j_{n-2}}.
\end{aligned}$$

An integration along the fibers of $X_{n-1} \rightarrow X_{n-2}$ then gives

$$\begin{aligned}
& \pi_{1,n-2}^* u_1^{j_1} \cdot \pi_{2,n-2}^* u_2^{j_2} \cdots u_{n-2}^{j_{n-2}} \cdot \pi_{0,n-2}^* c_1(X) \\
&= -(n-1) \cdot \sum_{s=1}^{n-2} \underbrace{\pi_{1,n-2}^* u_1^{j_1} \cdots \pi_{s,n-2}^* u_s^{j_s+1} \cdots u_{n-2}^{j_{n-2}}}_{=o(d^{n+1}) \text{ by the first part of the lemma}} \\
&\quad + o(d^{n+1})
\end{aligned}$$

and so $\pi_{1,n-2}^* u_1^{j_1} \cdot \pi_{2,n-2}^* u_2^{j_2} \cdots u_{n-2}^{j_{n-2}} \cdot \pi_{0,n-2}^* c_1(X) = o(d^{n+1})$.

To complete the proof, observe that – as before – relations (5) and (6) together with a completely similar degree argument give

$$\begin{aligned}
& \pi_{1,n-i}^* u_1^{j_1} \cdot \pi_{2,n-i}^* u_2^{j_2} \cdots u_{n-i}^{j_{n-i-1}} \cdot \pi_{0,n-i}^* c_1(X)^i \cdot u_{n-i}^n \\
&= -\pi_{1,n-i}^* u_1^{j_1} \cdot \pi_{2,n-i}^* u_2^{j_2} \cdots \pi_{n-i}^* u_{n-i-1}^{j_{n-i-1}} \cdot u_{n-i}^{n-1} \cdot \pi_{0,n-i}^* c_1(X)^{i+1} \\
&\quad - (n-1) u_{n-i}^{n-1} \cdot \sum_{s=1}^{n-i-1} \pi_{1,n-i}^* u_1^{j_1} \cdots \pi_{s,n-i}^* u_s^{j_s+1} \cdots \pi_{n-i}^* u_{n-i-1}^{j_{n-i-1}}.
\end{aligned}$$

But

$$\pi_{1,n-i}^* u_1^{j_1} \cdot \pi_{2,n-i}^* u_2^{j_2} \cdots u_{n-i}^{j_{n-i-1}} \cdot \pi_{0,n-i}^* c_1(X)^i \cdot u_{n-i}^n = o(d^{n+1})$$

by induction, and

$$\pi_{1,n-i}^* u_1^{j_1} \cdots \pi_{s,n-i}^* u_s^{j_s+1} \cdots \pi_{n-i}^* u_{n-i-1}^{j_{n-i-1}} = o(d^{n+1}),$$

$1 \leq s \leq n-i-1$, thanks to the first part of the lemma. \square

Lemma 3.3. *The coefficient of degree $n+1$ in d of $\pi_{1,n}^* u_1^n \cdot \pi_{2,n}^* u_2^n \cdots u_n^n$ is the same of the one of $(-1)^n c_1(X)^n$, that is 1.*

Proof. An explicit computation yields:

$$\begin{aligned}
\pi_{1,n}^* u_1^n \cdot \pi_{2,n}^* u_2^n \cdots u_n^n &\stackrel{(i)}{=} \pi_{1,n}^* u_1^n \cdot \pi_{2,n}^* u_2^n \cdots \pi_n^* u_{n-1}^n \left(-\pi_n^* c_1^{[n-1]} \cdot u_n^{n-1} \right. \\
&\quad \left. - \cdots - \pi_n^* c_{n-1}^{[n-1]} \cdot u_n - \pi_n^* c_n^{[n-1]} \right) \\
&\stackrel{(ii)}{=} -\pi_{1,n}^* u_1^n \cdot \pi_{2,n}^* u_2^n \cdots \pi_n^* u_{n-1}^n \cdot u_n^{n-1} \cdot \pi_n^* c_1^{[n-1]} \\
&\stackrel{(iii)}{=} -\pi_{1,n}^* u_1^n \cdot \pi_{2,n}^* u_2^n \cdots \pi_n^* u_{n-1}^n \cdot u_n^{n-1} \\
&\quad \cdot \pi_n^* \left(\pi_{0,n-1}^* c_1(X) + (n-1) \sum_{s=1}^{n-1} \pi_{s,n-1}^* u_s \right) \\
&\stackrel{(iv)}{=} -\pi_{1,n}^* u_1^n \cdot \pi_{2,n}^* u_2^n \cdots \pi_n^* u_{n-1}^n \cdot u_n^{n-1} \cdot \pi_{0,n}^* c_1(X) \\
&\quad + o(d^{n+1}) \\
&= \cdots \\
&\stackrel{(v)}{=} (-1)^n \pi_{0,k}^* c_1(X)^n \cdot \pi_{1,k}^* u_1^{n-1} \cdots u_n^{n-1} + o(d^{n+1}) \\
&\stackrel{(vi)}{=} (-1)^n c_1(X)^n + o(d^{n+1}).
\end{aligned}$$

Let us say a few words about the previous equalities. Equality (i) is just relation (5). Equality (ii) is true for degree reasons: $u_1^n \cdot u_2^n \cdots u_{n-1}^n \cdot c_l^{[n-1]}$, $l = 2, \dots, n$, “lives” on X_{n-1} and has total degree $n(n-1)+l$ which is strictly greater than $n+(n-1)(n-1) = \dim X_{n-1}$, so that $u_1^n \cdot u_2^n \cdots u_{n-1}^n \cdot c_l^{[n-1]} = 0$. Equality (iii) is just relation (6). Equality (iv) follows from the first part of Lemma 3.2: $u_1^n \cdots u_s^{n+1} \cdots u_{n-1}^n = o(d^{n+1})$. Equality (v) is obtained by applying repeatedly the second part of Lemma 3.2. Finally, equality (vi) is simply integration along the fibers. The lemma is proved. \square

Now, look at the coefficient of degree $n+1$ in d of the expression

$$\mathcal{O}_{X_n}(\mathbf{a})^{n^2} = (a_1 \pi_{1,n}^* u_1 + \cdots + a_n u_n)^{n^2},$$

where we consider the a_j ’s as variables: we claim that it is a non identically zero homogeneous polynomial of degree n^2 . To see this, we just observe that, thanks to Lemma 3.3, the coefficient of the monomial $a_1^n \cdots a_n^n$ is $(n^2)!/(n!)^n$.

Hence there exists an \mathbf{a} which satisfies the hypothesis of Corollary 3.1 for $k = n$, and Theorem 1.1 is proved.

4. THE LOGARITHMIC CASE AND PROOF OF THEOREM 1.2

Let $D \subset X$ be a simple normal crossing divisor in a compact complex manifold X , *i.e.* for each $x \in X$ there exist local holomorphic coordinates (z_1, \dots, z_n) for X , centered at x , such that locally $D = \{z_1 \cdots z_l = 0\}$, $0 \leq l \leq n$. Then $T_X^* \langle D \rangle$, the logarithmic cotangent space to X relative to D , is well defined and locally free: it is the subsheaf of the sheaf of meromorphic differential forms of the form

$$\sum_{j=1}^l f_j \frac{dz_j}{z_j} + \sum_{k=l+1}^n f_k dz_k,$$

where $f_i \in \mathcal{O}_{X,x}$ are germs of holomorphic functions in x and the local coordinates are chosen as above. Clearly, we have the following short exact sequence

$$(10) \quad 0 \rightarrow T_X^* \rightarrow T_X^* \langle D \rangle \rightarrow \mathcal{O}_D \rightarrow 0$$

and also $T_X^* \langle D \rangle|_{X \setminus D} = T_X^*$.

4.1. Chern classes computations. Here, we compute Chern classes for the logarithmic (co)tangent bundle of the pair (\mathbb{P}^n, D) , when D is a smooth projective hypersurface of degree $\deg D = d$. In this case (10) become

$$\begin{array}{ccccccc} & & & & 0 & & \\ & & & & \uparrow & & \\ 0 & \longrightarrow & T_{\mathbb{P}^n}^* & \longrightarrow & T_{\mathbb{P}^n}^* \langle D \rangle & \longrightarrow & \mathcal{O}_D \longrightarrow 0, \\ & & & & \uparrow & & \\ & & & & \mathcal{O}_{\mathbb{P}^n} & & \\ & & & & \uparrow & & \\ & & & & \mathcal{O}_{\mathbb{P}^n}(-D) & & \\ & & & & \uparrow & & \\ & & & & 0 & & \end{array}$$

where the vertical arrows are the usual locally free resolution of the structure sheaf of a divisor in \mathbb{P}^n ; then

$$c_\bullet(T_{\mathbb{P}^n}^* \langle D \rangle) = c_\bullet(T_{\mathbb{P}^n}^*) c_\bullet(\mathcal{O}_D) \quad \text{and} \quad 1 = c_\bullet(\mathcal{O}_{\mathbb{P}^n}(-D)) c_\bullet(\mathcal{O}_D),$$

so that if $h \in H^2(\mathbb{P}^n, \mathbb{Z})$ is the hyperplane class, we have

$$1 = (1 - dh) c_\bullet(\mathcal{O}_D)$$

and thus $c_\bullet(\mathcal{O}_D) = 1 + dh + (dh)^2 + \cdots + (dh)^n$. Now, recalling that $c_\bullet(T_{\mathbb{P}^n}) = (1 + h)^{n+1}$ and that, for a vector bundle E , $c_j(E^*) = (-1)^j c_j(E)$, we get the following:

Proposition 4.1. *Let $D \subset \mathbb{P}^n$ be a smooth hypersurface of degree $\deg D = d$. Then the Chern classes of the logarithmic tangent bundle $T_{\mathbb{P}^n} \langle D \rangle$ are given by*

$$(11) \quad c_j(T_{\mathbb{P}^n} \langle D \rangle) = (-1)^j h^j \sum_{k=0}^j (-1)^k \binom{n+1}{k} d^{j-k},$$

for $j = 1, \dots, n$.

4.2. Logarithmic jet bundles. Here, we recall the construction due to [De-L01] of logarithmic jet bundles, which is in fact completely analogous to the “standard” one.

So, we start with a triple (X, D, V) where (X, V) is a compact directed manifold and $D \subset X$ is a simple normal crossing divisor whose components $D_{(j)}$ are everywhere transversal to V (that is $T_{D_{(j)}} + V = T_X$ along $D_{(j)}$).

Let, as usual, $\mathcal{O}(V\langle D \rangle)$ be the (locally free in this setting) sheaf of germs of holomorphic vector fields which are tangent to each component of D .

Now, we define a sequence (X_k, D_k, V_k) of logarithmic k -jet bundles as in Subsection 2.2: if $(X_0, D_0, V_0) = (X, D, V\langle D \rangle)$, set $X_k = P(V_{k-1})$, $D_k = \pi_{0,k}^{-1}D$ and V_k is the set of logarithmic tangent vectors in $T_{X_k}\langle D_k \rangle$ which project onto the line defined by the tautological line bundle $\mathcal{O}_{X_k}(-1) \subset \pi_k^*V_{k-1}$.

In this case, the direct image formula of Theorem 2.1 becomes

$$(\pi_{0,k})_*\mathcal{O}_{X_k}(m) = \mathcal{O}(E_{k,m}V^*\langle D \rangle),$$

where $\mathcal{O}(E_{k,m}V^*\langle D \rangle)$ is the sheaf generated by all invariant polynomial differential operators in the derivatives of order $1, 2, \dots, k$ of the components f_1, \dots, f_r of a germ of holomorphic curve $f: (\mathbb{C}, 0) \rightarrow X \setminus D$ tangent to V , together with the extra functions $\log s_j(f)$ along the j -th components $D_{(j)}$ of D , where s_j is a local equation for $D_{(j)}$.

Then, as in the compact case, we have the following:

Theorem 4.1 ([De-L01]). *Assume that there exist integers $k, m > 0$ and an ample line bundle $A \rightarrow X$ such that*

$$H^0(X_k, \mathcal{O}_{X_k}(m) \otimes \pi_{0,k}^*A^{-1}) \simeq H^0(X, E_{k,m}V^*\langle D \rangle \otimes A^{-1})$$

has non zero sections $\sigma_1, \dots, \sigma_N$ and let $Z \subset X_k$ be the base locus of these sections. Then every entire holomorphic curve $f: \mathbb{C} \rightarrow X \setminus D$ tangent to V is such that $f_{[k]}(\mathbb{C}) \subset Z$.

Just like in the compact case, the locally free sheaf $\mathcal{O}(E_{k,m}V^*\langle D \rangle)$ arises naturally as a subsheaf of $\mathcal{J}_{k,m}\langle D \rangle$, of (non necessarily invariant) polynomial differential operators (cf. [De-L01], [Dem95]). Moreover, we can endow $\mathcal{J}_{k,m}V^*\langle D \rangle$ with a natural filtration with respect to the (weighted) degree such that the associated graded bundle is

$$\mathrm{Gr}^\bullet \mathcal{J}_{k,m}V^*\langle D \rangle = \bigoplus_{\ell_1 + 2\ell_2 + \dots + k\ell_k = m} S^{\ell_1}V^*\langle D \rangle \otimes \dots \otimes S^{\ell_k}V^*\langle D \rangle.$$

4.3. Strategy of the proof and logarithmic case. We begin with the following simple observation: the above construction of logarithmic jet bundles is, from the “relative” point of view, exactly the same as in the compact case.

This means that the short exact sequences which determine the relations on Chern classes and thus the relative structure of the cohomology algebra are, in the logarithmic case, the same as in Subsection 2.2.

Recall the main points of the proof in the compact case:

- For $\dim X = n$, go up to the n -th projectivized jet bundle, and find a (class of) relatively nef line bundle $\mathcal{O}_{X_n}(\mathbf{a}) \rightarrow X_n$, with a nontrivial morphism into $\mathcal{O}_{X_n}(m)$ for some large m .
- Write $\mathcal{O}_{X_n}(\mathbf{a})$ as the difference of two globally nef line bundle, namely

$$(\mathcal{O}_{X_k}(\mathbf{a}) \otimes \pi_{0,k}^*\mathcal{O}_X(2|\mathbf{a}|)) \otimes \pi_{0,k}^*\mathcal{O}_X(-2|\mathbf{a}|).$$

- Compute the “Morse” intersection $F^n - nF^{n-1} \cdot G$ for $\mathcal{O}_{X_n}(\mathbf{a})$ and show that, once expressed in term of the degree of X , the leading term is the same of $\mathcal{O}_{X_n}(\mathbf{a})^{n^2}$.

- Use the vanishing Theorem 2.4 to conclude that the term of maximal possible degree in $\mathcal{O}_{X_k}(\mathbf{a})^{n+k(n-1)}$ vanishes for $k < n$.
- Find a particular non-vanishing monomial in the variables \mathbf{a} , in the expression on maximal possible degree of $\mathcal{O}_{X_n}(\mathbf{a})^{n^2}$.

From this discussion, it follows that the only part which remains to be proved in the logarithmic case is an analogous of Theorem 2.4, all the rest being completely identical: this will be done in the next subsection.

To conclude the present paragraph, we just observe that the starting point to write $\mathcal{O}_{X_n}(\mathbf{a})$ as the difference of two globally nef line bundles is, for X a smooth projective hypersurface, that $T_X^* \otimes \mathcal{O}(2)$ is nef as a quotient of $T_{\mathbb{P}^{n+1}}^* \otimes \mathcal{O}(2)$. Thanks to the short exact sequence (10), this is the true also in the logarithmic case:

$$0 \rightarrow T_{\mathbb{P}^n}^* \otimes \mathcal{O}(2) \rightarrow T_{\mathbb{P}^n}^* \langle X \rangle \otimes \mathcal{O}(2) \rightarrow \mathcal{O}_X(2) \rightarrow 0,$$

and $T_{\mathbb{P}^n}^* \langle X \rangle \otimes \mathcal{O}(2)$ is nef as an extension of a nef vector bundle by a nef line bundle (compare with [Div08]).

4.4. Vanishing of global section of low order logarithmic jet differentials. We want to prove here the following vanishing theorem.

Theorem 4.2. *Let $D \subset \mathbb{P}^n$ be a smooth irreducible divisor of degree $\deg D = \delta$. Then*

$$H^0(\mathbb{P}^n, \mathcal{J}_{k,m} T_{\mathbb{P}^n}^* \langle D \rangle) = 0$$

for all $m \geq 1$ and $1 \leq k \leq n-1$, provided $\delta \geq 3$.

If we pass to the subbundle $E_{k,m} T_{\mathbb{P}^n}^* \langle D \rangle$ and add some negativity, we get an immediate corollary which is exactly what we need to conclude the proof of Theorem 1.2.

Corollary 4.1. *Let $D \subset \mathbb{P}^n$ be a smooth irreducible divisor of degree $\deg D = \delta$ and $A \rightarrow \mathbb{P}^n$ any ample line bundle. Then*

$$H^0(\mathbb{P}^n, E_{k,m} T_{\mathbb{P}^n}^* \langle D \rangle \otimes A^{-1}) = 0$$

for all $m \geq 1$ and $1 \leq k \leq n-1$, provided $\delta \geq 3$.

So, we begin recalling a vanishing theorem contained in [B-R90] for the twisted Schur powers of the cotangent bundle of a smooth complete intersection.

Let $Y = H_1 \cap \cdots \cap H_{N-n} \subset \mathbb{P}^N$ be an n -dimensional smooth complete intersection by the hypersurfaces $H_i \subset \mathbb{P}^N$, with $d_i = \deg H_i$.

Let $(\lambda) = (\lambda_1, \dots, \lambda_n)$ be a partition of the integer $r = \lambda_1 + \cdots + \lambda_n$, with $\lambda_1 \geq \cdots \geq \lambda_n \geq 0$, and $T_{(\lambda)}$ the associated Young tableau. Finally, let t_i be the number of cells inside the i -th column of $T_{(\lambda)}$ and set $t = \sum_{i=1}^{N-n} t_i$ (take $t_i = 0$ if $i > \text{length}(T)$).

Denote with $\Gamma^{(\lambda)} T_Y^*$ the irreducible representation of $\text{Gl}(T_Y^*)$ of highest weight (λ) (we refer the reader to [F-H91] for an excellent overview on representations of the general linear group and related things). Then we have the following:

Theorem 4.3 ([B-R90]). *If $p < r + \min\{\text{length}(T_{(\lambda)}), d_1 - 2, \dots, d_{N-n} - 2\}$ and $t < n$, then*

$$H^0(Y, \Gamma^{(\lambda)} T_Y^* \otimes \mathcal{O}_Y(p)) = 0.$$

In particular, if $Y \subset \mathbb{P}^{n+1}$ is a smooth projective hypersurface of degree $\deg Y = d$, then

$$H^0(Y, \Gamma^{(\lambda)} T_Y^* \otimes \mathcal{O}_Y(r+p)) = 0$$

if $\lambda_n = 0$ and $p < \min\{\lambda_1, d-2\}$.

From the above theorem, we deduce the following proposition which extends to all dimension a result of El Goul [EG03].

Proposition 4.2. *Let $D \subset \mathbb{P}^n$ a smooth hypersurface of degree $\deg D \geq 3$. Then*

$$H^0(\mathbb{P}^n, \Gamma^{(\lambda)} T_{\mathbb{P}^n}^* \langle D \rangle) = 0$$

for any non increasing n -tuple $(\lambda) = (\lambda_1, \dots, \lambda_n)$ with $\lambda_n = 0$.

Proof. Consider the standard ramified covering $\mathbb{P}^{n+1} \supset \tilde{D} \rightarrow \mathbb{P}^n$ associated to D : if D is given by the homogeneous equation $P(z_0, \dots, z_n) = 0$ of degree δ , then $\tilde{D} \subset \mathbb{P}^{n+1}$ is cut out by the single equation $z_{n+1}^\delta = P(z_0, \dots, z_n)$.

If we take pullbacks of logarithmic differential forms on \mathbb{P}^n , we obtain an injection $H^0(\mathbb{P}^n, T_{\mathbb{P}^n}^* \langle D \rangle) \hookrightarrow H^0(\tilde{D}, T_{\tilde{D}}^* \otimes \mathcal{O}_{\tilde{D}}(1))$. This is easily seen, as on \tilde{D} one has

$$\left. \frac{dP}{P} \right|_{\tilde{D}} = \left. \frac{dz_{n+1}^\delta}{z_{n+1}^\delta} \right|_{\tilde{D}} = \delta \left. \frac{dz_{n+1}}{z_{n+1}} \right|_{\tilde{D}},$$

so that pullbacks of logarithmic forms downstairs give rise to forms with one simple pole along the hyperplane section $\{z_{n+1} = 0\} \cap \tilde{D}$.

Now, we just have to apply, given the weight λ , the Schur functors to the injection $H^0(\mathbb{P}^n, T_{\mathbb{P}^n}^* \langle D \rangle) \hookrightarrow H^0(\tilde{D}, T_{\tilde{D}}^* \otimes \mathcal{O}_{\tilde{D}}(1))$, in order to obtain the new injection

$$H^0(\mathbb{P}^n, \Gamma^{(\lambda)} T_{\mathbb{P}^n}^* \langle D \rangle) \hookrightarrow H^0(\tilde{D}, \Gamma^{(\lambda)} T_{\tilde{D}}^* \otimes \mathcal{O}_{\tilde{D}}(|\lambda|)),$$

where $|\lambda| = \lambda_1 + \dots + \lambda_n$. The proposition follows from Theorem 4.3 (with $r = |\lambda|$ and $p = 0$). \square

4.4.1. *End of the proof of the vanishing.* To conclude the proof of Theorem 4.2, we just need to exclude — using the same strategy as in [Div08] — among the irreducible $\mathrm{Gl}(T_{\mathbb{P}^n}^* \langle D \rangle)$ -representations of the bundle $\mathcal{J}_{k,m} T_{\mathbb{P}^n}^* \langle D \rangle$ with $k < n$, Schur powers of the form

$$\Gamma^{(\lambda_1, \dots, \lambda_n)} T_{\mathbb{P}^n}^* \langle D \rangle$$

with $\lambda_n \neq 0$. This is possible thanks to the following elementary lemma.

Lemma 4.1. *Let V be a complex vector space of dimension n and $\lambda = (\lambda_1, \dots, \lambda_n)$ such that $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0$. Then*

$$\Gamma^{(\lambda)} V \otimes S^m V \simeq \bigoplus_{\mu} \Gamma^{(\mu)} V$$

as $\mathrm{Gl}(V)$ -representations, the sum being over all μ whose Young diagram $T_{(\mu)}$ is obtained by adding m boxes to the Young diagram $T_{(\lambda)}$ of λ , with no two in the same column.

Proof. This follows immediately by Pieri's formula, see e.g. [F-H91]. \square

Note that this implies that among all the irreducible $\mathrm{Gl}(V)$ -representations of $S^l V \otimes S^m V$, we cannot find terms of type $\Gamma^{(\lambda_1, \dots, \lambda_n)} V$ with $\lambda_i > 0$ for $i > 2$ (they are all of type $\Gamma^{(l+m-j, j, 0, \dots, 0)} V$ for $j = 0, \dots, \min\{m, l\}$).

Thus, by induction on the number of factor in the tensor product of symmetric powers, we easily find:

Corollary 4.2. *If $k \leq n$, then we have a direct sum decomposition into irreducible $\mathrm{Gl}(V)$ -representations*

$$S^{\ell_1} V \otimes S^{\ell_2} V \otimes \dots \otimes S^{\ell_k} V = \bigoplus_{\lambda} \nu_{\lambda} \Gamma^{(\lambda)} V,$$

where $\nu_{\lambda} \neq 0$ only if $\lambda = (\lambda_1, \dots, \lambda_n)$ is such that $\lambda_i = 0$ for $i > k$.

So, in our hypotheses, the composition series of $\mathcal{J}_{k,m} T_{\mathbb{P}^n}^* \langle D \rangle$ has vanishing H^0 group, and Theorem 4.2 is proved.

4.5. Effective Results for the Existence of Logarithmic Jet Differentials in Low Dimension. The get the effective results announced in the statements of Theorem 1.2, we just compute the algebraic holomorphic Morse inequalities, for $\mathbf{a} = (2 \cdot 3^{n-2}, \dots, 6, 2, 1) \in \mathbb{N}^n$. Hence we get an explicit polynomial in the variable d , which has positive leading coefficient, and we compute its largest positive root.

All this is done by implementing a quite simple code on GP/PARI CALCULATOR Version 2.3.2. The computation complexity blows-up rapidly and, starting from dimension 6, our computers were not able to achieve any result in a finite time.

Remark 2. Although very natural, we don't know if the weight

$$\mathbf{a} = (2 \cdot 3^{n-2}, \dots, 6, 2, 1) \in \mathbb{N}^n$$

we utilize is the best possible.

Here is the code.

```
/*scratch variable*/
X

/*main formal variables*/
c=[c1,c2,c3,c4,c5,c6,c7,c8,c9] /*Chern classes of V<D> on P^n*/
u=[u1,u2,u3,u4,u5,u6,u7,u8,u9] /*Chern classes of OXk(1)*/
v=[v1,v2,v3,v4,v5,v6,v7,v8,v9] /*Chern classes of Vk on Xk*/
w=[w1,w2,w3,w4,w5,w6,w7,w8,w9] /*formal variables*/
e=[0,0,0,0,0,0,0,0,0] /*empty array for logarithmic Chern classes*/
q=[0,0,0,0,0,0,0,0,0] /*empty array for Chern equations*/

/*main*/
Calcul(dim,order)=
{
local(j,n,N);
n=dim;
r=dim;
k=order;
```

```

N=n+k*(r-1);
H(n);
Chern();
B=2*h*3^(k-1);
A=B+u[k];
for(j=1,k-1,A=A+2*3^(k-j-1)*u[j]);
R=Reduc((A-N*B)*A^(N-1));
C=Integ(R);
print("Calculation for order ", k, " jets on logarithmic projective
", n, "-space");
print("Line bundle A= ", A);
print("Line bundle B= ", B);
print("Chern class of A^", N, "-", N, "*A^", N-1, "*B :");
print(C);
E=Eval(C);
print("Evaluation for degree d logarithmic projective ", n,"-space:");
print(E)
}

```

```

/*compute Chern relations*/
Chern()=
{
local(j,s,t);
q[1]=X^r; for(j=1,r,q[1]=q[1]+c[j]*X^(r-j));
for(s=1,r,v[s]=c[s]);
for(s=r+1,9,v[s]=0);
for(t=1,k-1,
for(s=1,r,w[s]=v[s]+(binomial(r,s)-binomial(r,s-1))*u[t]^s;
for(j=1,s-1,w[s]=w[s]+
(binomial(r-j,s-j)-binomial(r-j,s-j-1))*v[j]*u[t]^(s-j)));
for(s=1,r,v[s]=w[s]);
q[t+1]=X^r; for(j=1,r,q[t+1]=q[t+1]+v[j]*X^(r-j)))
}

```

```

/*reduction to Chern classes of (P^n,D)*/
Reduc(p)=
{
local(j,a);
a=p;
for(j=0,k-1,
a=subst(a,u[k-j],X);
a=subst(lift(Mod(a,q[k-j])),X,u[k-j]));
a
}

```

```

/*integration along fibers*/
Integ(p)=
{

```

```

local(j,a);
a=p;
for(j=0,k-1,
a=polcoeff(a,r-1,u[k-j]));
a
}

/*compute Chern classes of degree d logarithmic projective n-space*/
H(n)=
{
local(j,s);
for(s=1,n,
e[s]=d^s;
for(j=1,s,e[s]=e[s]+(-1)^j*(d)^(s-j)*binomial(n+1,j));
e[s]=(-1)^s*e[s])
}

/*evaluation in terms of the degree*/
Eval(p)=
{
local(a,s);
a=p;
for(s=1,r,a=subst(a,c[s],e[s]));
subst(a,h,1)*d
}

```

REFERENCES

- [B-R90] Brückmann P., Rackwitz, H.-G.: *T-Symmetrical Tensor Forms on Complete Intersections*. Math. Ann. **288** (1990), no. 4, 627–635.
- [Dem95] Demailly, J.-P.: *Algebraic Criteria for Kobayashi Hyperbolic Projective Varieties and Jet Differentials*. Algebraic geometry—Santa Cruz 1995, 285–360, Proc. Sympos. Pure Math., 62, Part 2, Amer. Math. Soc., Providence, RI, 1997.
- [Dem00] Demailly, J.-P.: *Multiplier Ideal Sheaves and Analytic Methods in Algebraic Geometry*. School on vanishing Theorems and Effective Results in Algebraic Geometry (Trieste, 2000), 1–148, ICTP Lect. Notes, 5, Abdus Salam Int. Cent. Theoret. Phys., Trieste, 2001.
- [D-EG00] Demailly, J.-P., El Goul, J.: *Hyperbolicity of Generic Surfaces of High Degree in Projective 3-Space*. Amer. J. Math **122** (2000), no. 3, 515–546.
- [De-L01] Dethloff, G.-E., Lu, Steven S.-Y.: *Logarithmic Jet Bundles and Applications*. Osaka J. Math. **38** (2001), no. 1, 185–237.
- [Div08] Diverio, S.: *Differential Equations on Complex Projective Hypersurfaces of Low Dimension*. Compos. Math. **144** (2008), no. 4, 920–932.
- [EG03] El Goul, J.: *Logarithmic Jets and Hyperbolicity*. Osaka J. Math. **40** (2003), no. 2, 469–491.
- [F-H91] Fulton, W., Harris, J.: *Representation Theory: A First Course*. Graduate Texts in Mathematics, 129. Readings in Mathematics. Springer-Verlag, New York, 1991. xvi+551 pp.
- [G-G79] Green, M., Griffiths, P.: *Two Applications of Algebraic Geometry to Entire Holomorphic Mappings*. The Chern Symposium 1979 (Proc. Internat. Sympos., Berkeley, Calif., 1979), pp. 41–74, Springer, New York-Berlin, 1980.

- [Kob70] Kobayashi S.: *Hyperbolic Manifolds and Holomorphic Mappings*. Marcel Dekker, Inc., New York 1970 ix+148 pp.
- [Rou06] Rousseau, E: *Équations Diffrentielles sur les Hypersurfaces de \mathbb{P}^4* . J. Math. Pures Appl. (9) **86** (2006), no. 4, 322–341.
- [Siu04] Siu, Y.-T.: *Hyperbolicity in Complex Geometry*. The legacy of Niel Henrik Abel, 543–566, Springer, Berlin, 2004.
- [Tra95] Trapani, S.: *Numerical criteria for the positivity of the difference of ample divisors*. Math. Z. **219** (1995), no. 3, 387–401.

ISTITUTO “GUIDO CASTELNUOVO”, SAPIENZA UNIVERSITÀ DI ROMA
E-mail address: `diverio@mat.uniroma1.it`

INSTITUT “FOURIER”, UNIVERSITÉ DE GRENOBLE I
E-mail address: `sdiverio@fourier.ujf-grenoble.fr`